Use the half-angle identities to find the exact value of each.

22) $\sin 67.5°$ $\frac{135°}{2} = 67.5°$

$\sin 67.5° = \sin\frac{135°}{2} = \sqrt{\frac{1-\cos 135°}{2}}$

$= \sqrt{\frac{1-\frac{-\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}}$

$= \sqrt{\frac{2+\sqrt{2}}{2}\cdot\frac{1}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{4}} = \boxed{\frac{\sqrt{2+\sqrt{2}}}{2}}$

23) $\cos 75°$ $\frac{150°}{2} = 75°$

$\cos 75° = \cos\left(\frac{150°}{2}\right) = \sqrt{\frac{1+\cos 150°}{2}}$

$= \sqrt{\frac{1+\frac{-\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2}{2}-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}$

$= \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{\sqrt{4}} = \boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}$

24) $\tan 255°$ $\frac{510°}{2} = 255°$

$= \frac{1-\cos 510°}{\sin 510°} \overset{-360°}{=} \frac{1-\cos 150°}{\sin 150°} = \frac{1-\frac{-\sqrt{3}}{2}}{1/2} = \frac{\frac{2}{2}-\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\frac{2+\sqrt{3}}{2}}{\frac{1}{2}} = \frac{2+\sqrt{3}}{2}\cdot\frac{2}{1} = \boxed{2+\sqrt{3}}$

Verify each identity.

25) $\frac{1-\cos 2x}{\cos^2 x} = 2\tan^2 x$ GIVEN

$\frac{1-(1-2\sin^2 x)}{\cos^2 x} = 2\tan^2 x$ DOUBLE-ANGLE ID.

$\frac{2\sin^2 x}{\cos^2 x} = 2\tan^2 x$ SUBTRACT (DISTRIBUTE THE −)

$2\tan^2 x = 2\tan^2 x$ ■ QUOTIENT ID.

26) $\dfrac{1+\cos 2x}{\cos x} = 2\cos x$ GIVEN

$\dfrac{1+2\cos^2 x-1}{\cos x} = 2\cos x$ DOUBLE-ANGLE ID.

$\dfrac{2\cos^2 x}{\cos x} = 2\cos x$ SUBTRACT/ADD

$2\cos x = 2\cos x$ ■ REDUCE/DIVIDE.

27) $\dfrac{1-\cos 2x}{\tan^2 x} = 1+\cos 2x$ GIVEN

$\dfrac{1-(1-2\sin^2 x)}{\tan^2 x} = 1+\cos 2x$ DOUBLE-ANGLE ID.

$\dfrac{2\sin^2 x}{\tan^2 x} = 1+\cos 2x$ DISTRIBUTE THE - (AND SUBTRACT)

$\dfrac{2\sin^2 x}{1} \cdot \dfrac{\cos^2 x}{\sin^2 x} = 1+\cos 2x$ QUOTIENT ID.

$2\cos^2 x = 1+\cos 2x$ DIVIDE/REDUCE

$1+\cos 2x = 1+\cos 2x$ ■ DOUBLE-ANGLE ID.

28) $1 + \cos 2x + \sin 2x = 2\cos x \cdot (\sin x + \cos x)$

$1 + \cos 2x + 2\sin x\cos x = 2\cos x(\sin x + \cos x)$ GIVEN

$1 + 2\cos^2 x - 1 + 2\sin x\cos x = 2\cos x(\sin x + \cos x)$ DOUBLE-ANGLE ID.

$2\cos^2 x + 2\sin x\cos x = 2\cos x(\sin x + \cos x)$ DOUBLE-ANGLE ID.

$2\cos x(\cos x + \sin x) = 2\cos x(\sin x + \cos x)$ ADD|SUBTRACT

FACTOR (GCF)

Solve each equation for $0 \le \theta < 2\pi$.

29) $2\sec^2\theta = -2\sqrt{3}\sec\theta - \sec^2\theta$

$3\sec^2\theta + 2\sqrt{3}\sec\theta = 0$

$\sec\theta(3\sec\theta + 2\sqrt{3}) = 0$

$\sec\theta = 0$

$\frac{1}{\cos\theta} = \frac{0}{1}$

$\cos\theta = \emptyset$

NO SOLUTION

$3\sec\theta + 2\sqrt{3} = 0$

$3\sec\theta = -2\sqrt{3}$

$\sec\theta = \frac{-2\sqrt{3}}{3}$

$\frac{1}{\cos\theta} = \frac{-2\sqrt{3}}{3}$

$\cos\theta = \frac{3}{-2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$\cos\theta = \frac{3\sqrt{3}}{-6}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

$\theta = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

#1: $\theta = \frac{5\pi}{6}$

#2: $2\pi - \frac{5\pi}{6} = \frac{7\pi}{6}$

30) $\sqrt{3}\sec\theta\cot\theta - \sec\theta - 2\cot\theta = -2\cot\theta$

$\sqrt{3}\sec\theta\cot\theta - \sec\theta = 0$

$\sec\theta(\sqrt{3}\cot\theta - 1) = 0$

$\sec\theta = 0$

$\frac{1}{\cos\theta} = \frac{0}{1}$

$\cos\theta = \emptyset$

NO SOLUTION

$\sqrt{3}\cot\theta - 1 = 0$

$\sqrt{3}\cot\theta = 1$

$\cot\theta = \frac{1}{\sqrt{3}}$

$\frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}$

$\tan\theta = \sqrt{3}$

$\theta = \tan^{-1}(\sqrt{3})$

#1: $\theta = \frac{\pi}{3}$

#2: $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$

31) $-\cos^2\theta = 2\sin\theta - 2$

$0 = \cos^2\theta + 2\sin\theta - 2$

$0 = 1 - \sin^2\theta + 2\sin\theta - 2$

$0 = -\sin^2\theta + 2\sin\theta - 1$ $*-1$

$0 = \sin^2\theta + 2\sin\theta + 1$

$0 = (\sin\theta - 1)(\sin\theta - 1)$

$0 = (\sin\theta - 1)^2$

$0 = \sin\theta - 1$

$1 = \sin\theta$

$\sin^{-1}(1) = \theta$

#1: $\frac{\pi}{2} = \theta$

#2: $\pi - \frac{\pi}{2} =$ SAME

32) $-\sin\theta + \cos^2\theta = \sin^2\theta$

$-\sin\theta + 1 - \sin^2\theta = \sin^2\theta$

$0 = 2\sin^2\theta + \sin\theta - 1$

$0 = (2\sin\theta - 1)(\sin\theta + 1)$

$2\sin\theta - 1 = 0$

$\sin\theta = 1/2$

$\theta = \sin^{-1}(1/2)$

#1: $\theta = \pi/6$

#2: $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\sin\theta + 1 = 0$

$\sin\theta =$

$\theta = \sin^{-1}(-1)$

#1: $\theta = \frac{-\pi}{2} + 2\pi = \frac{3\pi}{2}$

#2: $\pi - \frac{-\pi}{2} =$ SAME

33) $2\cos^2\theta = 1 - \cos 2\theta$

$2\cos^2\theta = 1 - (2\cos^2\theta - 1)$

$2\cos^2\theta = 1 - 2\cos^2\theta + 1$

$4\cos^2\theta - 2 = 0$

$4\cos^2\theta = 2$

$\cos^2\theta = \frac{1}{2}$

$\cos\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{1}}{\sqrt{2}} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$

$\cos\theta = \pm\frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$

#1: $\theta = \frac{\pi}{4}$

#2: $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$\cos\theta = \frac{-\sqrt{2}}{2}$

$\theta = \cos^{-1}\left(\frac{-\sqrt{2}}{2}\right)$

#1: $\theta = \frac{3\pi}{4}$

#2: $2\pi - \frac{3\pi}{4} = \frac{5\pi}{4}$

*34) $-\sin^2 2\theta = 2\sin^2\theta - 2\sin^2 2\theta$

$\sin^2 2\theta - 2\sin^2\theta = 0$

$4\sin^2\theta\cos^2\theta - 2\sin^2\theta = 0$

$2\sin^2\theta(2\cos^2\theta - 1) = 0$

$2\sin^2\theta = 0$

$\sin^2\theta = 0$

$\sin\theta = 0$

$\theta = \sin^{-1}(0)$

#1: $\theta = 0$

#2: $\pi - 0 = \pi$

$2\cos^2\theta - 1 = 0$

$2\cos^2\theta = 1$

$\cos^2\theta = 1/2$

$\cos\theta = \pm\sqrt{1/2}$

$\cos\theta = \pm\frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$

#1: $\theta = \frac{\pi}{4}$

#2: $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$\cos\theta = \frac{-\sqrt{2}}{2}$

$\theta = \cos^{-1}\left(\frac{-\sqrt{2}}{2}\right)$

#1: $\theta = \frac{3\pi}{4}$

#2: $2\pi - \frac{3\pi}{4} = \frac{5\pi}{4}$