Unit 6

Chapter 13

Sequences, Series, & Limits

13-1, 13-2, & 13-6

Arithmetic & Geometric Sequences,

Recursive Definitions,

& Sigma Notation

Definition of *Sequence*

In math, the word ***sequence*** means that a collection is listed in a specific order, so it has a first member, second member, and so on.

These "members" of the sequence are called ***terms***.

The function's…

first term $= f(1) = a_1$

second term $= f(2) = a_2$

third term $= f(3) = a_3$

n^{th} term $= f(n) = a_n$

Definition of *Sequence*

- The function values $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ are the ***terms*** of the sequence.
- An ***infinite sequence*** is a function whose domain is the set of positive integers.
- If the domain of the function consists of the first *n* positive integers only, the sequence is a ***finite sequence***.
- In an ***arithmetic sequence***, each term has a common difference (*d*).

  Example: 1, 6, 11, 16, 21, 26, … $d = 5$

- In a ***geometric sequence***, each term has a common ratio (*r*).

  Example: 3, 12, 48, 192, … $r = 4$

Explicit Formula

Because listing out just the first few terms of a sequence isn't enough to define the sequence, a formula must be provided.

An ***explicit formula*** allows you to quickly find the n^{th} term of the sequence without knowing any of the terms.

Examples:

$a_n = 3n - 2$ $\qquad$ $a_n = 3 + (-1)^n$ $\qquad$ $a_n = \frac{(-1)^n}{2n-1}$

Example 1:

Find the first four terms of the sequences given by:

a) $a_n = 3n - 2$

1st term: $a_1 = 3(1) - 2 = 1$

2nd term: $a_2 = 3(2) - 2 = 4$

3rd term: $a_3 = 3(3) - 2 = 7$

4th term: $a_4 = 3(4) - 2 = 10$

Is this sequence arithmetic, geometric, or neither?

arithmetic: the common difference is 3

Example 1:

Find the first four terms of the sequences given by:

b) $a_n = 3 + (-1)^n$

1st term: $a_1 = 3 + (-1)^1 = 2$

2nd term: $a_2 = 3 + (-1)^2 = 4$

3rd term: $a_3 = 3 + (-1)^3 = 2$

4th term: $a_4 = 3 + (-1)^4 = 4$

Is this sequence arithmetic, geometric, or neither?

neither

Example 2:

Find the first five terms of the sequence given by $a_n = \frac{(-1)^n}{2n-1}$

$$a_1 = \frac{(-1)^1}{2(1) - 1} = -1$$

$$a_2 = \frac{(-1)^2}{2(2) - 1} = \frac{1}{3}$$

$$a_3 = \frac{(-1)^3}{2(3) - 1} = -\frac{1}{5}$$

$$a_4 = \frac{(-1)^4}{2(4) - 1} = \frac{1}{7}$$

$$a_5 = \frac{(-1)^5}{2(5) - 1} = -\frac{1}{9}$$

Recursive Formula

A ***recursive formula*** requires you to be given one or more of the first few terms.

All other terms of the sequence are defined using previous terms.

$$\begin{cases} a_1 = first\ term \\ a_n = a_{n-1}\ math\ stuff \end{cases}$$

Example 4:

Find the first four terms for the sequence: $\begin{cases} a_1 = -3 \\ a_n = a_{n-1} \times 3 \text{ where } n \geq 2 \end{cases}$

$$-3, -9, -27, -81$$

Example 5:

Find the first four terms for the sequence: $\begin{cases} a_1 = -3 \\ a_n = a_{n-1} + n \text{ where } n \geq 2 \end{cases}$

$-3, -1, 2, 6$

Example 6:

Find d and the explicit and recursive formulas for each arithmetic sequence.

$7, 11, 15, 19, \ldots$

$d = 4$

Explicit: $a_n = 4n + 3$

Recursive: $\begin{cases} a_1 = 7 \\ a_n = a_{n-1} + 4 \end{cases}$

Example 7: Find d and the explicit and recursive formulas for each arithmetic sequence.

$2, -3, -8, -13, \ldots$

$d = -5$

Explicit: $a_n = -5n + 7$

Recursive: $\begin{cases} a_1 = 2 \\ a_n = a_{n-1} - 5 \end{cases}$

Example 8: Find d and the explicit and recursive formulas for each arithmetic sequence.

$1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \ldots \longrightarrow \frac{4}{4}, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}, \ldots$

$$d = \frac{1}{4}$$

Explicit: $a_n = \frac{1}{4}n + \frac{3}{4}$

Recursive:
$$\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + \frac{1}{4} \end{cases}$$

Example 9: Find r and the explicit and recursive formulas for each geometric sequence.

$2, 4, 8, 16, \ldots$

$r = 2$

Explicit: $a_n = 2^{n-1} \times 2$

Recursive: $\begin{cases} a_1 = 2 \\ a_n = a_{n-1} \times 2 \end{cases}$

Example 10: Find r and the explicit and recursive formulas for each geometric sequence.

$12, 36, 108, 324, \ldots$

$r = 3$

Explicit: $a_n = 3^{n-1} \times 12$

Recursive: $\begin{cases} a_1 = 12 \\ a_n = a_{n-1} \times 3 \end{cases}$

Example 11: Find r and the explicit and recursive formulas for each geometric sequence.

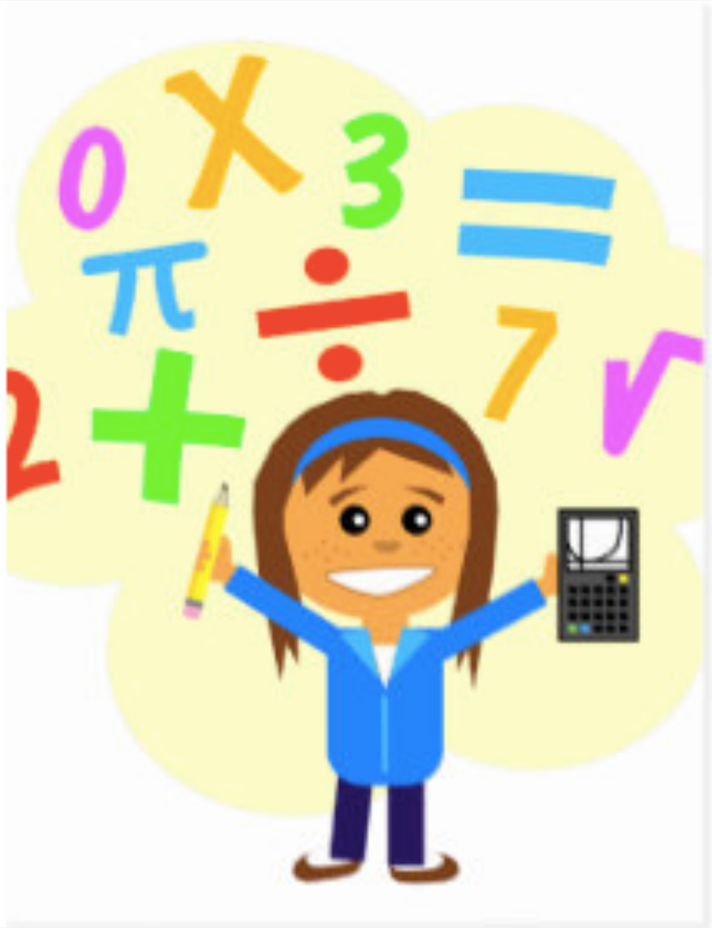
$$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots$$

$$r = -\frac{1}{3}$$

Explicit: $a_n = -\frac{1}{3}^{n-1} \times -\frac{1}{3}$

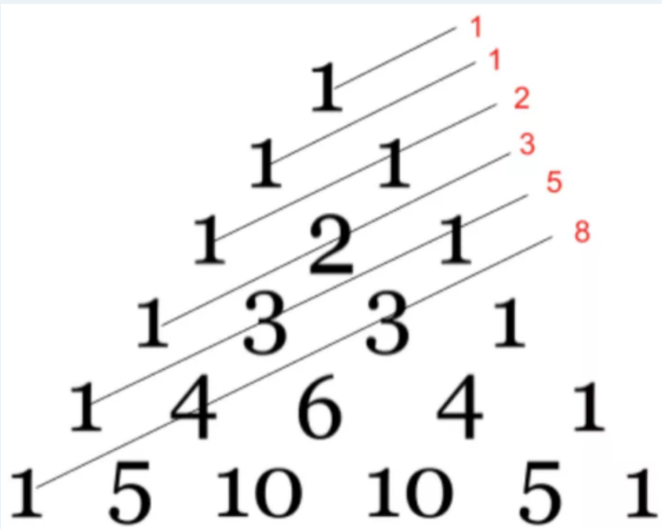
Recursive: $\begin{cases} a_1 = -\frac{1}{3} \\ a_n = a_{n-1} \times -\frac{1}{3} \end{cases}$

Now you try...



The Fibonacci Sequence

$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n-2} + a_{n-1}, \text{where } n \geq 3 \end{cases}$$



$a_1 = 1$

$a_2 = 1$

$a_3 = 2 = a_1 + a_2$

$a_4 = 3 = a_2 + a_3$

$a_5 = 5 = a_3 + a_4$

$a_6 = 8 = a_4 + a_5$

$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1},$ where $n \geq 3$

Factorials

If n is a positive integer, n factorial is defined as:

$$n! = n * (n-1) * (n-2) * ... * 1$$

Examples:

$$5! = 5 * 4 * 3 * 2 * 1$$

$$9! = 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1$$

Factorials

If n is a positive integer, n factorial is defined as:

$$n! = n * (n-1) * (n-2) *. . . * 1$$

As a special case, zero factorial is defined as $0! = 1$

★ Note that

$$2n! = 2(n!) = 2(1 * 2 * 3 * \ldots * n)$$

$$(2n)! = (1 * 2 * 3 * \ldots * 2n)$$

Example 12

Find & plot the first 5 terms of the sequence given by $a_n = \frac{2^{(n-1)}}{(n-1)!}$

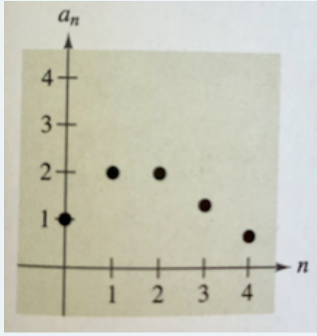
$a_1 = \frac{2^{(1-1)}}{(1-1)!} = \frac{2^0}{0!} = \frac{1}{1} = 1$

$a_2 = \frac{2^{(2-1)}}{(2-1)!} = \frac{2^1}{1!} = \frac{2}{1} = 2$

$a_3 = \frac{2^{(3-1)}}{(3-1)!} = \frac{2^2}{2!} = \frac{4}{2} = 2$

$a_4 = \frac{2^{(4-1)}}{(4-1)!} = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3}$

$a_5 = \frac{2^{(5-1)}}{(5-1)!} = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3}$



Example 13

Evaluate each factorial expression without a calculator:

a) $$\frac{8!}{2!\ *\ 6!} = \frac{8*7*\cancel{6*5*4*3*2*1}}{2*1*\cancel{6*5*4*3*2*1}} = \frac{56}{2} = 28$$

b) $$\frac{2!\ *6!}{3!\ *5!} = \frac{\cancel{2*1}*6*\cancel{5*4*3*2*1}}{3*\cancel{2*1}*\cancel{5*4*3*2*1}} = \frac{6}{3} = 2$$

c) $$\frac{n!}{(n-1)!} = \frac{n*\cancel{(n-1)*(n-2)*(n-3)*\ldots}}{\cancel{(n-1)*(n-2)*(n-3)*\ldots}} = n$$

Summation Notation

Σ

Capital
sigma

σ

lower-case
sigma

Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

Where i is called the **index** of summation, n is the **upper limit** of summation, and 1 is the **lower limit** of summation.

Example 1

Find each sum.

a) $$\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5) = 45$$

b) $$\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2) = 90$$

c) $$\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} \approx 2.71828$$

Practice:

Page 621-622 (in other book)

#2-10 evens, 25-28, 37-40, 41, 42, 44, 46,
55-58, 64-70 evens, 71-79 odds